A GENERALIZATION OF k-COHEN-MACAULAY COMPLEXES

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Abstract. For a positive integer k and a non-negative integer t a class of simplicial complexes, to be denoted by k-CM_t, is introduced. This class generalizes two notions for simplicial complexes: being k-Cohen-Macaulay and k-Buchsbaum. In analogy with the Cohen-Macaulay and Buchsbaum complexes, we give some characterizations of $\mathrm{CM}_t(=1\text{-}\mathrm{CM}_t)$ complexes, in terms of vanishing of some homologies of its links and, in terms of vanishing of some relative singular homologies of the geometric realization of the complex and its punctured space. We show that a complex is k-CM_t if and only if the links of its nonempty faces are k-CM_{t-1}. We prove that for an integer $s \leq d$, the $(d-s-1)$-skeleton of a $(d-1)$-dimensional k-CM_t complex is $(k+s)$-CM_t. This result generalizes Hibi's result for Cohen-Macaulay complexes and Miyazaki's result for Buchsbaum complexes.

1. Introduction

Let K be a fixed field. The Stanley-Reisner ring of a simplicial complex over K provides a "bridge" to transfer properties in commutative algebra such as being Cohen-Macaulay or Buchsbaum into simplicial complexes. The main advantage in the study of simplicial complexes is the interplay between their algebraic, combinatorial, homological and topological properties. Stanley's book [16] is a suitable reference for a comprehensive introduction to the subject. The aim of this paper is to introduce and develop basic properties of a new class of simplicial complexes, called k-CM_t complexes, which generalizes two notions for simplicial complexes: being k-Cohen-Macaulay, and being k-Buchsbaum.

In Section 2, we introduce CM_t complexes and discuss their basic properties. We show that for a pure simplicial complex Δ of dimension $(d-1)$ the following are equivalent, (see Theorems 2.5 and 2.7):

(i) Δ is CM_t.
(ii) $\tilde{H}_i(\mathrm{lk}_{\Delta}(\sigma); K) = 0$ for all $\sigma \in \Delta$ with $\#\sigma \geq t$ and $i < d - \#\sigma - 1$.
(iii) $H_i(|\Delta|, |\Delta| \setminus p; K) = 0$ for all $p \in |\Delta| \setminus |\Delta_{t-2}|$ and all $i < d-1$, where $|\Delta|$ is the geometric realization of Δ and Δ_{t-2} is the $(t-2)$-skeleton of Δ.

In Section 3, k-CM_t complexes are introduced and some of their basic properties are studied. We show that a complex is k-CM_t if and only if the links of its nonempty faces are k-CM_{t-1} (see Proposition 3.6). We consider a simplicial complex Δ and certain faces $\sigma_1, \cdots, \sigma_\ell$ of Δ such that

(i) $\sigma_i \cup \sigma_j \notin \Delta$ if $i \neq j$.
(ii) If $\Delta_1 = \{\tau \in \Delta | \tau \not\supseteq \sigma_i \text{ for all } i\}$ then $\dim \Delta_1 < \dim \Delta$.

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In [7] Hibi showed that that Δ_1 is 2-Cohen-Macaulay of dimension $(\dim \Delta - 1)$ provided that Δ is Cohen-Macaulay and $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen-Macaulay for all i. In [10] Miyazaki extended this result for Buchsbaumness by showing that if Δ is a Buchsbaum complex of dimension $d-1$, and $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen-Macaulay for all i, then Δ_1 is 2-Buchsbaum. We prove that a similar result is valid for CM_t complexes (see Theorem 3.8). This leads to prove that for an integer $s \leq d$, the $(d-s-1)$-skeleton of a $(d-1)$-dimensional k-CM_t complex is $(k+s)$-CM_t (see Corollary 3.10). This generalizes a result of Terai and Hibi [18] (also see [3]) which asserts that the 1-skeleton of of a simplicial $(d-1)$-sphere with $d \geq 2$ is d-connected. It also generalizes a result of Hibi [7] (see [10, Introduction]) which says that if Δ is a Cohen-Macaulay complex of dimension $d-1$, then the $(d-2)$-skeleton of Δ is 2-Cohen-Macaulay.

2. The CM_t simplicial complexes

In this section we introduce CM_t complexes and discuss their basic properties. We give some characterizations of CM_t complexes, in terms of vanishing of some homologies of its links (see Theorem 2.5), and, in terms of vanishing of some relative singular homologies of the geometric realization of the complex and its punctured space (see Theorem 2.7). First recall that for any face σ of the simplicial complex Δ, the link of σ is defined as follows:

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta | \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}.$$

Definition 2.1. *Let K be a field, Δ a simplicial complex of dimension $(d-1)$ over K. Let t be an integer $0 \leq t \leq d-1$. Then Δ is called CM_t over K if Δ is pure and $lk_\Delta(\sigma)$ is Cohen-Macaulay over K for any $\sigma \in \Delta$ with $\#\sigma \geq t$.*

We will adopt the convention that for $t \leq 0$, CM_t means CM_0. Note that from the results by Reisner [13] and Schenzel [15] it follows that CM_0 is the same as Cohen-Macaulayness and CM_1 is identical with Buchsbaum property. It is also clear that for any $j \geq i$, CM_i implies CM_j.

Example 2.2. *Let Δ be the union of two $(d-1)$-simplicies that intersect in a $(t-2)$-dimensional face where $1 \leq t \leq d-1$. Then Δ is a CM_t complex which is not a CM_{t-1} complex. In fact, if Γ is a finite union of $(d-1)$-simplicies where any two of them intersect in a face of dimension at most $t-2$, then Γ is a CM_t complex, and if at least two of the simplicies have a $(t-2)$-dimensional face in common, then Γ is not CM_{t-1}. These include simplicial complexes corresponding to the transversal monomial ideals which happen to have linear resolutions* [21].

It is known that the links of a Cohen-Macaulay simplicial complex are also Cohen-Macaulay, see [8]. As the first result of this section we show that a similar property holds for CM_t complexes. In the rest of this paper we freely use the following fact:
For all $\sigma \in \Delta$ and all $\tau \in \mathrm{lk}_\Delta(\sigma)$,

$$\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma)}(\tau) = \mathrm{lk}_\Delta(\sigma \cup \tau).$$

Lemma 2.3. *Let Δ be a simplicial complex. Then the following are equivalent.*

(i) *Δ is a CM_t complex.*
(ii) *Δ is pure and $lk_\Delta(\{x\})$ is CM_{t-1} for all $\{x\} \in \Delta$.*

Proof. (i)$\Rightarrow$(ii). Let $\{x\} \in \Delta$ and $\tau \in \mathrm{lk}_{\Delta}(\{x\})$ with $\#\tau \geq t-1$. Since Δ is CM_t and $\#(\{x\} \cup \tau) \geq t$ we see that $\mathrm{lk}_{\mathrm{lk}_{\Delta}(\{x\})}(\tau) = \mathrm{lk}_{\Delta}(\{x\} \cup \tau)$ is Cohen-Macaulay. In addition, since Δ is pure it follows that $\mathrm{lk}_{\Delta}(\{x\})$ is pure for all $x \in \Delta$.

(ii)$\Rightarrow$(i). Let $\sigma \in \Delta$ with $\#\sigma \geq t$. Let $x \in \sigma$, $\tau = \sigma \setminus \{x\}$. Then $\#\tau \geq t-1$ and $\mathrm{lk}_{\Delta}\sigma = \mathrm{lk}_{\Delta}(\{x\} \cup \tau) = \mathrm{lk}_{\mathrm{lk}_{\Delta}(\{x\})}(\tau)$ is Cohen-Macaulay.

□

We recall Reisner's characterization of Cohen-Macaulay simplicial complexes [13, Theorem 1].

Theorem 2.4. *Let Δ be a simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is Cohen-Macaulay over K.*
(ii) *$\tilde{H}_i(lk_{\Delta}(\sigma); K) = 0$ for any $\sigma \in \Delta$ and all $i < dim\,(lk_{\Delta}(\sigma))$.*

In analogy with the above result, the following theorem provides equivalent conditions for CM_t complexes.

Theorem 2.5. *Let Δ be a simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is CM_t over K;*
(ii) *Δ is pure and $\tilde{H}_i(lk_{\Delta}(\sigma); K) = 0$ for all $\sigma \in \Delta$ with $\#\sigma \geq t$ and $i < d - \#\sigma - 1$.*

Proof. (i)$\Rightarrow$(ii). Suppose that Δ is CM_t over K. Then Δ is pure and $\mathrm{lk}_{\Delta}(\sigma)$ is Cohen-Macaulay for all $\sigma \in \Delta$ with $\#\sigma \geq t$. Therefore, $\tilde{H}_i(\mathrm{lk}_{\mathrm{lk}_{\Delta}(\sigma)}(\tau); K) = 0$ for all $\tau \in \mathrm{lk}_{\Delta}(\sigma)$ and all $i < \dim(\mathrm{lk}_{\mathrm{lk}_{\Delta}(\sigma)}(\tau))$. In particular, for $\tau = \varnothing$, $\mathrm{lk}_{\mathrm{lk}_{\Delta}(\sigma)}(\varnothing) = \mathrm{lk}_{\Delta}(\sigma)$ and we have $\tilde{H}_i(\mathrm{lk}_{\Delta}(\sigma); K) = 0$ for all $i < \dim(\mathrm{lk}_{\Delta}(\sigma)) \leq d - \#\sigma - 1$.

(ii)$\Rightarrow$(i). We use induction on t. Use [15, Theorem 3.2] for the case $t = 1$. Assume that the assertion holds for $t-1$. Let $\{x\} \in \Delta$, $\tau \in \mathrm{lk}_{\Delta}\{x\}$ with $\#\tau \geq t-1$. Then by purity of Δ, $\dim \mathrm{lk}_{\Delta}\{x\} = d-2$. But $\tau \cup \{x\} \in \Delta$ and hence by (ii), $\tilde{H}_i(\mathrm{lk}_{\Delta}(\tau \cup \{x\}); K) = 0$ for all $i < d - \#\tau - 2$. This implies that $\tilde{H}_i(\mathrm{lk}_{\mathrm{lk}_{\Delta}(\{x\})}(\tau; K) = 0$ for all $\tau \in \mathrm{lk}_{\Delta}\{x\}$ with $\#\tau \geq t-1$, and all $i < d-1-\#\tau-1$. By induction hypothesis $\mathrm{lk}_{\Delta}\{x\}$ is CM_{t-1} for all $\{x\} \in \Delta$. Now by Lemma 2.3 we are done.

□

We state a result due to Munkres [11, Corollary 3.4] which states that Cohen-Macaulayness is a topological property.

Theorem 2.6. *Let Δ be a pure simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is Cohen-Macaulay over K.*
(ii) *$\tilde{H}_i(|\Delta|; K) = 0 = H_i(|\Delta|, |\Delta| \setminus p; K)$ for all $p \in |\Delta|$ and all $i < d-1$, where $|\Delta|$ is the geometric realization of Δ.*

The following theorem may lead one to believe that the property CM_t is also a topological invariant.

Theorem 2.7. *Let Δ be a pure simplicial complex of dimension $(d-1)$. Then the following are equivalent:*

(i) *Δ is* CM_t *over* K*;*
(ii) $H_i(|\Delta|, |\Delta| \setminus p; K) = 0$ *for all* $p \in |\Delta| \setminus |\Delta_{t-2}|$ *and all* $i < d-1$*, where* Δ_{t-2} *is the* $(t-2)$*-skeleton of* Δ *and* $|\Delta_{t-2}|$ *is induced from a fixed geometric realization of* Δ*.*

Proof. First note that by Theorem 2.5, Δ is CM_t if and only if $\tilde{H}_i(\mathrm{lk}_\Delta(\sigma); K) = 0$ for all $\sigma \in \Delta$ with $\#\sigma \geq t$ and all $i < d - \#\sigma - 1$. Now by [11, Lemma 3.3], for any interior point p of σ we have

$$H_i(|\Delta|, |\Delta| \setminus p; K) \cong \tilde{H}_{i-\#\sigma}(\mathrm{lk}_\Delta(\sigma); K).$$

Therefore, $H_i(|\Delta|, |\Delta| \setminus p; K) = 0$ for any $\sigma \in \Delta$ with $\#\sigma \geq t$ and any interior point of σ, and, any $i < d-1$ if and only if $\tilde{H}_i(\mathrm{lk}_\Delta(\sigma); K) = 0$ for all $\sigma \in \Delta$ with $\#\sigma \geq t$ and $i < d - \#\sigma - 1$. But the set of such points is precisely $|\Delta| \setminus |\Delta_{t-2}|$ when some geometric realization is fixed.

□

Let Δ and Δ' be a two simplicial complex whose vertex sets are disjoint. The simplicial join $\Delta * \Delta'$ is defined to be the simplicial complex whose faces are of the form $\sigma \cup \sigma'$ where $\sigma \in \Delta$ and $\sigma' \in \Delta'$.

The algebraic and combinatorial properties of the simplicial join $\Delta * \Delta'$ through the properties of Δ and Δ' have been studied by a number of authors (see [5], [6], [12], and [1]). For instance, in [6], Fröberg used the (graded) K-algebra isomorphism $K[\Delta * \Delta'] \simeq K[\Delta] \otimes_K K[\Delta']$, and proved that the tensor product of two graded K-algebras is Cohen-Macaulay (resp. Gorenstein) if and only if both of them are Cohen-Macaulay (resp. Gorenstein). One can see that the simplicial join of the triangulation of a cylinder (which is Buchsbaum [17, Example II.2.13(i)]) with a simplicial complex with only one vertex (which is Cohen-Macaulay [17, Example II.2.14(ii)] and so Buchsbaum) is not Buchsbaum. In [14] it is shown that $\Delta * \Delta'$ is Buchsbaum (over K) if and only if Δ and Δ' are Cohen-Macaulay (over K). Therefore, it is natural to ask about Δ and Δ' when $\Delta * \Delta'$ is CM_t. At present these authors do not know the answer.

3. The k-CM_t simplicial complexes

In this section k-CM_t complexes are introduced and some of their basic properties are given. We show that a complex is k-CM_t if and only if the links of its nonempty faces are k-CM_{t-1} (see Proposition 3.6). The main result of this section is Theorem 3.8 which states that certain subcomplex of a CM_t complex is 2-CM_t. This leads to prove that for an integer $s \leq d$, the $(d-s-1)$-skeleton of a $(d-1)$-dimensional k-CM_t complex is $(k+s)$-CM_t (see Corollary 3.10).

Definition 3.1. *Let K be a field. For positive integer k and non-negative integer t, a simplicial complex Δ with vertex set V is called k-CM_t of dimension r over K if for any subset W of V (including $\varnothing$) with $\#W < k$, $\Delta_{V\setminus W}$ is CM_t of dimension r over K. The complex Δ is k-CM_t over K if Δ is k-CM_t of some dimension r over K.*

Note that for any $\ell \leq k$, k-CM_t implies ℓ-CM_t. In particular, any k-CM_t is CM_t.

In the rest of this paper we will often need the following lemma [10, Lemma 2.3].

Lemma 3.2. *Let Δ be a simplicial complex with vertex set V. Let $W \subseteq V$ and let σ be a face in Δ. If $W \cap \sigma = \varnothing$, then $lk_{\Delta_{V\setminus W}}(\sigma) = lk_\Delta(\sigma)_{V\setminus W}$.*

Remark 3.3. *One may call Δ a $(k\text{-CM})_t$ complex if for all $\sigma \in \Delta$ with $\#\sigma \geq t$, $lk_\Delta(\sigma)$ is k-Cohen-Macaulay. But this is the same as k-CM_t property because both properties require that for $W \subset V$ with $\#W < k$, $lk_{\Delta_{V\setminus W}}(\sigma) = lk_\Delta(\sigma)_{V\setminus W}$ is Cohen-Macaulay.*

Lemma 3.4. *Let Δ be a k-CM_t complex and let $\sigma \in \Delta$ be an arbitrary face with $\#\sigma = s$. Then $lk_\Delta(\sigma)$ is k-CM_{t-s}.*

Proof. Let V_1 be the vertex set of $\mathrm{lk}_\Delta(\sigma)$ and consider $W \subset V_1$ with $\#W < k$. We need to show that, $(\mathrm{lk}_\Delta(\sigma))_{V_1\setminus W}$ is CM_{t-s}. Observe that since $\sigma \cap W = \varnothing$, $\mathrm{lk}_\Delta(\sigma)_{V_1\setminus W} = \mathrm{lk}_\Delta(\sigma)_{V\setminus W} = \mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)$. Put $\Gamma = \mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)$ and let $\tau \in \Gamma$ with $\#\tau \geq t-s$. Then $\#(\sigma\cup\tau) \geq t$ and $\mathrm{lk}_\Gamma(\tau) = \mathrm{lk}_{\Delta_{V\setminus W}}(\sigma\cup\tau)$, which is Cohen-Macaulay by assumption. $\square$

Corollary 3.5. *Let Δ be a k-Buchsbaum (k-CM_2) complex and let $\sigma \in \Delta$ be a non-empty face. Then $lk_\Delta(\sigma)$ is k-Cohen-Macaulay (resp. k-Buchsbaum).*

Proposition 3.6. *Let Δ be a pure complex of dimension $(d-1)$ with vertex set V. Then for all positive integers k and t the following are equivalent:*

(i) *Δ is k-CM_t.*
(ii) *For any non-empty face σ in Δ, $lk_\Delta(\sigma)$ is a k-CM_{t-1}.*

Proof. (i)$\Rightarrow$ (ii): Use Lemma 3.4.

(ii)$\Rightarrow$ (i): For any subset W of V with $\#W < k$, we need to show that $\Delta_{V\setminus W}$ is CM_t of dimension $d-1$. Let $\sigma \in \Delta_{V\setminus W}$ with $\#\sigma \geq t$. Then $\mathrm{lk}_{\Delta_{V\setminus W}}(\sigma) = (\mathrm{lk}_\Delta(\sigma))_{V\setminus W}$. Since $\mathrm{lk}_\Delta(\sigma)$ is a k-CM_{t-1} we have that $\mathrm{lk}_{\Delta_{V\setminus W}}(\sigma)$ is Cohen-Macaulay.

Now we show that $\Delta_{V\setminus W}$ is pure of dimension $(d-1)$. Let τ be an arbitrary facet in $\Delta_{V\setminus W}$. Since $\mathrm{lk}_\Delta(\tau)$ is a k-CM_{t-1} complex, we have

$$\dim(\mathrm{lk}_\Delta(\tau)_{V\setminus W}) = \dim(\mathrm{lk}_\Delta(\tau)).$$

On the other hand since Δ is pure, we have $\dim(\mathrm{lk}_\Delta(\tau)) = d - \#\tau - 1$. In addition,

$$\dim(\mathrm{lk}_\Delta(\tau)_{V\setminus W}) = \dim(\mathrm{lk}_{\Delta_{V\setminus W}}(\tau)) = \dim(\{\varnothing\}) = -1.$$

Therefore, we have $\dim(\tau) = d-1$.

$\square$

Corollary 3.7. *(see* [10, Lemma 4.2]*) Let Δ be a pure complex of dimension $(d-1)$ with vertex set V. Then for all positive integers k the following are equivalent:*

(i) *Δ is k-Buchsbaum.*
(ii) *For any non-empty face σ in Δ, $lk_\Delta(\sigma)$ is a k-Cohen-Macaulay complex.*

Now we are ready to give one of the main results of this paper which generalizes results due to Hibi [7] and Miyazaki [10].

Let Δ a simplicial complex and let $\sigma_1, \cdots, \sigma_\ell$ be faces of Δ such that

(i) $\sigma_i \cup \sigma_j \notin \Delta$ if $i \neq j$.
(ii) If $\Delta_1 = \{\tau \in \Delta | \tau \not\supseteq \sigma_i \text{ for all } i\}$ then $\dim \Delta_i < \dim \Delta$.

In [7] Hibi showed that Δ_1 is 2-Cohen-Macaulay of dimension $(\dim \Delta - 1)$ provided that $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen-Macaulay for all i. In [10] Miyazaki extended this result for Buchsbaumness by showing that if Δ is a Buchsbaum complex of dimension $d-1$, and $\mathrm{lk}_\Delta(\sigma_i)$ is 2-Cohen-Macaulay for all i, then Δ_1 is 2-Buchsbaum.

Now it is natural to ask whether the similar result is valid for CM_t complexes. In the following result we give an affirmative answer to this question.

Theorem 3.8. *Let Δ be a CM_t complex and let $\sigma_1, \cdots, \sigma_\ell$ be faces of Δ satisfying the above conditions (i) and (ii). If $\mathrm{lk}_\Delta(\sigma_i)$ is 2-CM_{t-1} for all i, then Δ_1 is 2-CM_t complex of dimension $(\dim \Delta - 1)$.*

Proof. We use induction on t. If $t = 0, 1$ the assertion hold by [7] and [10, Theorem 7.4]. Assume that the assertion holds for $t - 1$. By Lemma 3.6 we need to show that Δ_1 is pure and for any non-empty face τ in Δ_1, $\mathrm{lk}_{\Delta_1}(\tau)$ is 2-CM_{t-1}. By [10, Lemma 7.2], Δ_1 is pure. Let τ be a non-empty face in Δ_1. We may reorder σ_i's such that $\sigma_i \cup \tau \in \Delta$ if and only if $i \leq s$. Then

$$\begin{aligned} \mathrm{lk}_{\Delta_1}(\tau) = & \quad \{\sigma \in \Delta | \sigma \cup \tau \in \Delta_1, \sigma \cap \tau = \varnothing\} \\ = & \quad \{\sigma \in \Delta | \sigma \cup \tau \in \Delta, \sigma \cap \tau = \varnothing, \sigma \cup \tau \not\supseteq \sigma_i (1 \leq i \leq \ell)\} \\ = & \quad \{\sigma \in \Delta | \sigma \cup \tau \in \Delta, \sigma \cap \tau = \varnothing, \sigma \not\supseteq \tau_i (1 \leq i \leq s)\} \end{aligned}$$

where $\tau_i = \sigma_i - \tau$ for $1 \leq i \leq s$. Thus if we put $\Gamma = \mathrm{lk}_\Delta(\tau)$ then

$$\mathrm{lk}_{\Delta_1}(\tau) = \{\sigma \in \Gamma | \sigma \not\supseteq \tau_i (1 \leq i \leq s)\}.$$

On the other hand,

$$\mathrm{lk}_\Gamma(\tau_i) = \mathrm{lk}_\Delta(\tau \cup \tau_i) = \mathrm{lk}_{\mathrm{lk}_\Delta(\sigma_i)}(\tau - \sigma_i).$$

By assumption $\mathrm{lk}_\Delta(\sigma_i)$ is 2-CM_{t-1}. Then by Lemma 3.4, $\mathrm{lk}_{\mathrm{lk}_\Delta(\sigma_i)}(\tau - \sigma_i)$ is 2-CM_{t-2} and hence $\mathrm{lk}_\Gamma(\tau_i)$ is 2-CM_{t-2}. Applying the induction hypothesis for Γ and $\tau_1, \cdots, \tau_s$ it follows that $\mathrm{lk}_{\Delta_1}(\tau)$ is 2-CM_{t-1}. Since τ is an arbitrary non-empty face of Δ_1, by Lemma 3.6, it follows that Δ_1 is a 2-CM_t complex of dimension $(\dim \Delta - 1)$. □

The condition on $\mathrm{lk}_\Delta(\sigma_i)$ in the above theorem can not be weakened in the sense that one can not replace CM_{t-1} by CM_t for these links. This can be seen in the following example.

Example 3.9. *(see* [10, Example 7.5]*) If*

$$\Delta_1 =< \{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{1,5\}, \{4,5\} >,$$

which has dimension 1, and $\Delta_2 =< \{x, y\} >$, then $\Delta = \Delta_1 * \Delta_2$ is Cohen-Macaulay. If we put $\sigma_1 = \{x, y\}$, $t = 1$, then $\mathrm{lk}_\Delta(\sigma_1) = \Delta_1$ is a 2-Buchsbaum complex and $\Delta \setminus \sigma_1 = \Delta_1 * < \{x\}, \{y\} >$. So $\mathrm{lk}_{\Delta \setminus \sigma_1}(\{x\}) = \Delta_1$ is not 2-Cohen-Macaulay and we see that $\Delta \setminus \sigma_1$ is not 2-Buchsbaum.

Corollary 3.10. *Let Δ be a k-CM_t complex of dimension $(d-1)$. If $s \leq d$ and Δ' is the $(d-s-1)$-skeleton of Δ, then Δ' is $(k+s)$-CM_t.*

Proof. We may assume $s = 1$. Let V be the vertex set of Δ and W a subset of V such that $0 < \#W < k+1$. If we take $x \in W$ and put $W' = W \setminus \{x\}$, then $\Delta_{V \setminus W'}$ is CM_t of dimension $(d-1)$ by assumption. On the other hand since

$$\Delta'_{V \setminus W'} = \{\sigma \in \Delta | \dim(\sigma) < d-1, \sigma \cap W' = \varnothing\}$$

and this is equal to the $(d-2)$-skeleton of $\Delta_{V \setminus W'}$, by Theorem 3.8, $(\Delta')_{V \setminus W'}$ is 2-$\mathrm{CM_t}$ of dimension $(d-2)$. So $(\Delta')_{V \setminus W' - \{x\}} = (\Delta')_{V \setminus W}$ is a CM_t complex of dimension $(d-2)$.

□

Remark 3.11. *The above corollary generalizes a result of Terai and Hibi* [18] *(see also* [3]*) which states that the 1-skeleton of a simplicial $(d-1)$-sphere with $d \geq 2$ is d-connected (topological). This is just due to the fact that a simplicial $(d-1)$-sphere is 2-Cohen-Macaulay and $(d-1)$-Cohen-Macaulayness implies $(d-1)$-connectedness. This corollary also generalizes a result of Hibi* [7] *(see* [10, Introduction]*) which says that if Δ is a Cohen-Macaulay complex of dimension $d-1$, then the $(d-2)$-skeleton of Δ is 2-Cohen-Macaulay.*

Example 3.12. *If Γ is a finite union of $(d-1)$-simplicies where any two of them intersect in a face of dimension at most $t-2$ and Λ is the $(d-2)$-skeleton of Γ, then Λ is 2-CM_t. If at least two facets in Γ intersect in a $(t-2)$-dimensional face, then Λ is not 2-CM_{t-1}.*

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References

1. C. A. Athanasiadis, V. Welker, Buchsbaum* complexes, arXiv: 0909.1931v2.
2. D. Barnette, *Graph theorems for manifolds*, Israel J. Math. **16** (1973), 6372.
3. K. Baclawski, *Cohen-Macaulay connectivity and geometric lattices*, European J. Combinatorics **3** (1982), 293–305.
4. A. Björner, M. Wachs, *Shellable nonpure complexes and posets, I*, Trans. Amer. Math. Soc. **348** (1996), 1299–1327.
5. A. Björner, M. Wachs and V. Welker, *On sequentially Cohen-Macaulay complexes and posets*, Israel J. Math. **169** (2009), 295–316.
6. R. Fröberg, *A note on the Stanley-Reisner ring of a join and of a suspension.* Manuscripta Math. **60** (1988), no. 1, 89–91.
7. T. Hibi, *Level rings and algebras with straightening laws*, J. of Algebra, **117** (1988), 343–362.
8. M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes, Ring Theory II, Proc. of the second Oklahoma Conf. (B. R. MacDonald and R. Morris, ed.*, Lect. Notes in Pure and Appl. Math., No. 26, Dekker, New York, 1977, 171–223.
9. M. Miyazaki, *Characterizations of Buchsbaum complexes*, Manuscripta Math. **63** (1989), 245–254.
10. M. Miyazaki, *On 2-Buchsbaum complexes*, J. Math. Kyoto Univ. **30** (1990), 367–392.
11. J. Munkres, *Topological results in combinatorics*, Michigan Math. J. **31** (1984), 113–128.
12. S. J. Provan and L. J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), 576–594.
13. G.A. Reisner, *Cohen-Macaulay quotients of polynomial rings*, Adv. Math. **21**(1976), 30–49.
14. H. Sabzrou, M. Tousi, S. Yassemi, *Simplicial join via tensor products*, Manuscripta Math. **126** (2008), 255-272.
15. P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. **178** (1981), 125–142.
16. R. Stanley, *Combinatorics and Commutative Algebra,* Second Edition, Birkhauser, Boston, 1995.
17. J. Stückrad and W. Vogel, *Buchsbaum rings and applications. An interaction between algebra, geometry, and topology.* Mathematical Monographs, Berlin, 1986.
18. N. Terai, T. Hibi, *Finite free resolutions and 1-skeletons of simplicial complexes*, J. Alg. Combin. **6** (1997), 8993.

19. N. Terai, *Alexander duality in Stanley-Reisner rings*, in "Affine Algebraic Geometry (T. Hibi, ed.)", Osaka University Press, Osaka 2007, 449–462.
20. J.W. Walker, *Topology and combinatorics of ordered sets*, Ph.D Thesis, MIT, 1981.
21. R. Zaare-Nahandi and R. Zaare-Nahandi, The minimal free resolution of a class of square-free monomial ideals, Journal of Pure and Applied Algebra 189 (2004) 263-278.

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